Eigenvalues and Eigenvectors

* I assume you have reviewed basic matrix algebra and know how to add, subtract, multiply matrices and multiply matrices by a scalar.

The Identity matrix -

Define I_n = the $n \times n$ identity matrix $\begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & \cdots & \cdots & 1 \end{pmatrix}$ which is the $n \times n$ identity matrix with all diagonal elements equal to 1 and all off-diagonal elements equal to 0.

Then, if multiplication is defined, i.e. I is $n \times n$, A is $n \times m$, then $IA = A$. Similarly, if A is $m \times n$ and I is $n \times n$ then $AI = A$. Importantly, if A is square $n \times n$ then $IA = AI = A$.

Solution of a system -

A square linear system in n unknowns $x_1, x_2, \ldots, x_n$ with constant coefficients is of the form -

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= b_2 \\ a_{31}x_1 + a_{32}x_2 + \ldots + a_{3n}x_n &= b_3 \\ \vdots \qquad & \quad \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n &= b_n \end{aligned}$$

Where a_{ij} = constant for all ij.

This can be written in matrix form as $Ax = b$ where

$$A = \begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & & & \vdots \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{pmatrix}, \quad x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \quad b = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$$

Recall that a square matrix A has an inverse, A^{-1}, if

$$A^{-1}A = AA^{-1} = I.$$

So if the matrix A has an inverse A^{-1} then the equation $Ax = b$ has a solution:

$$A^{-1}Ax = A^{-1}b$$
$$Ix = A^{-1}b$$

and $x = A^{-1}b$ is the $n \times 1$ solution $\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$

alternatively, you should recall solution by Cramer's rule. I will prove this only for the 2×2 system –

1) $ax_1 + bx_2 = e$
2) $cx_1 + dx_2 = f$

where $a, b, \ldots, f$ are constant.

Note this is equivalent to the matrix equation $AX = Y$ where

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad Y = \begin{pmatrix} e \\ f \end{pmatrix}$$

from equation 1 –

$$x_2 = \frac{e - ax_1}{b}$$

substituting into 2 we find

$$cx_1 + d\left(\frac{e - ax_1}{b}\right) = f$$

$$bcx_1 - adx_1 = bf - de$$

or $$x_1 = \frac{de - bf}{ad - bc}$$

and $$x_2 = \frac{e - a\left(\frac{de - bf}{ad - bc}\right)}{b}$$

$$= \frac{ade - bce - ade + abf}{b(ad - bc)}$$

$$= \frac{af - ce}{ad - bc}$$

Now note that $x_1 = \dfrac{\begin{vmatrix} e & b \\ f & d \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}}$ and $x_2 = \dfrac{\begin{vmatrix} a & e \\ c & f \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}}$

This solution can be written in the form

$$x_1 = \frac{D_1}{D}, \quad x_2 = \frac{D_2}{D}$$

where $D = \det(A)$ = determinant of A and

$$D_1 = \begin{vmatrix} e & b \\ f & d \end{vmatrix}, \quad D_2 = \begin{vmatrix} a & e \\ c & f \end{vmatrix}$$

This extends inductively to the solution of an $n \times n$ system –

$Ax = b$ where $A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & & & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}, \; x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \; b = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$

and $x_1 = \frac{D_1}{D}, \; x_2 = \frac{D_2}{D}, \dots, x_n = \frac{D_n}{D}$

where $D = \det(A)$ and D_i = determinant of the matrix A with the i^{th} column of A replaced by b.

So we have the result we need – namely that the system $Ax = b$ has no solution if $D = \det(A) = 0$. But if A^{-1} exists then $x = A^{-1}b$ so if no solution exists, i.e. $\det(A) = 0$, then A^{-1} must not exist. In this case, A is called <u>singular</u> or <u>non-invertible</u>.

The result is –

$$\boxed{\text{The system } Ax = b \text{ has no solution if } A \text{ is singular, i.e. } A^{-1} \text{ does not exist, if } \det(A) = 0.}$$

Eigenvalues and eigenvectors –

<u>Def</u>: If A is an $n \times n$ matrix, x is an $n \times 1$ vector, and $Ax = \lambda x$, where λ = constant, then x is an <u>eigenvector</u> of A with corresponding <u>eigenvalue</u> λ. You may sometimes see the terminology <u>characteristic vector</u> and <u>characteristic value</u>.

<u>Ex</u>: Suppose $A = \begin{pmatrix} 4 & -2 \\ 1 & 1 \end{pmatrix}, \; x = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$,

Then $Ax = \begin{pmatrix} 4 & -2 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 6 \\ 3 \end{pmatrix} = 3\begin{pmatrix} 2 \\ 1 \end{pmatrix}$

So x is an eigenvector of A with eigenvalue 3.

Note that eigenvectors are not unique because if x is an eigenvector of A, i.e. $Ax = \lambda x$, then if c = constant –

$A(cx) = cAx = c\lambda x = \lambda(cx)$ and cx is also an eigenvector for any c with the same eigenvalue λ.

From the result above we have a method of finding eigenvectors —

Suppose $Ax = \lambda x$

then $\quad Ax = \lambda I x$ where I is the identity matrix

$$Ax - \lambda I x = 0$$

$$(A - \lambda I)x = 0$$

So if x is not the trivial solution $x = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$ then it must be true that $A - \lambda I$ is singular, i.e. $(A-\lambda I)^{-1}$ does not exist, because if $(A - \lambda I)$ is invertible then $(A - \lambda I)^{-1}(A - \lambda I)x = Ix = x = 0$, and the only solution is $x = 0 = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$.

Ex: Find the eigenvectors and corresponding eigenvalues of $A = \begin{pmatrix} 4 & -2 \\ 1 & 1 \end{pmatrix}$.

Suppose $Ax = \lambda x$, i.e. x is an eigenvector of A with corresponding eigenvalue λ. Then —

$$Ax = \lambda I x$$

$$Ax - \lambda I x = 0$$

$$(A - \lambda I)x = 0$$

So if $x \neq 0$ then $A - \lambda I$ is singular and $\det(A - \lambda I) = |A - \lambda I| = 0$.

$$|A - \lambda I| = \left| \begin{pmatrix} 4 & -2 \\ 1 & 1 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right| = \left| \begin{pmatrix} 4 & -2 \\ 1 & 1 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \right|$$

$$= \left| \begin{pmatrix} 4-\lambda & -2 \\ 1 & 1-\lambda \end{pmatrix} \right|$$

$$= (4-\lambda)(1-\lambda) - (-2) = 0$$

or $\quad \lambda^2 - 5\lambda + 6 = 0$

$$(\lambda - 2)(\lambda - 3) = 0$$

and $\quad \lambda = 2,\ \lambda = 3$ are the eigenvalues of A.

Now we find the corresponding eigenvectors individually for each eigenvalue from the above expression $(A - \lambda I)x = 0$

Suppose $\lambda = 2$ and $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ is the corresponding eigenvector.

Then $(A - \lambda I)x = \left\{ \begin{pmatrix} 4 & -2 \\ 1 & 1 \end{pmatrix} - \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \right\} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2 & -2 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

Multiplying, we find

$$\begin{pmatrix} 2x_1 - 2x_2 \\ x_1 - x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

notice this gives 2 equivalent equations because there are an infinite # of eigenvectors differing by a constant!

or $x_1 - x_2 = 0$

and $x_2 = x_1$

So any vector $\begin{pmatrix} x_1 \\ x_1 \end{pmatrix}$ or $c\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ where c = constant is an eigenvector with eigenvalue $\lambda = 2$.

Now suppose $\lambda = 3$. Then

$$(A - 3I)x = 0$$

$$\left\{\begin{pmatrix} 4 & -2 \\ 1 & 1 \end{pmatrix} - \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}\right\}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -2 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

or $x_1 - 2x_2 = 0$

so $x_1 = 2x_2$ and any vector of the form $\begin{pmatrix} 2x_2 \\ x_2 \end{pmatrix}$, or $c\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ (for any c = constant) is an eigenvector with eigenvalue $\lambda = 3$. Note this is the example given above.

Note that this method extends to finding eigenvectors of any size $n \times n$ matrix A but we will be primarily interested in the 2×2 case.
